Int J Theor Phys (2010) 49: 1065–1072
DOI 10.1007/s10773-010-0285-6

Entangled Husimi Distribution and Complex Wavelet Transformation

Li-Yun Hu · Hong-Yi Fan

Received: 5 December 2009 / Accepted: 21 February 2010 / Published online: 6 March 2010

Abstract Similar in spirit to the preceding work (Int. J. Theor. Phys. 48:1539, 2009) where the relationship between wavelet transformation and Husimi distribution function is revealed, we study this kind of relationship to the entangled case. We find that the optical complex wavelet transformation can be used to study the entangled Husimi distribution function in phase space theory of quantum optics. We prove that, up to a Gaussian function, the entangled Husimi distribution function of a two-mode quantum state $|\psi\rangle$ is just the modulus square of the complex wavelet transform of $e^{-|\eta|^2/2}$ with $\psi(\eta)$ being the mother wavelet.

Keywords Complex wavelet transformation · Entangled Husimi distribution · IWOP technique

1 Introduction

Phase space technique has proved very effective in various branches of physics. Studying distribution functions of density operator ρ in phase space has been a major topic in quantum optics and quantum statistical physics. Among various phase space distributions the Wigner function $F_w(q, p)$ [1–5] is the most popularly used. But the Wigner distribution function itself is not a probability distribution because it may take negative value. To avoid this situation, the Husimi distribution function $F_h(q', p')$ is introduced [6], which is defined in a manner that guarantees it non-negative. On the other hand, since 1980s the optical wavelet

L.-Y. Hu (✉)
College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang 330022, China
e-mail: hlyun2008@126.com

H.-Y. Fan
Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, China

transformation has been developed which can overcome some shortcomings of usual classical Fourier transformation and therefore has been widely used in signal analysis and detection [7–10]. In the preceding paper [11], in single-mode case we have employed the optical wavelet transformation to study the Husimi distribution function, and proved that, up to a Gaussian function $e^{-\frac{p^2}{\kappa}}$, the Husimi distribution function of a quantum state $|\psi\rangle$ is just the modulus square of the wavelet transform of $e^{-x^2/2}$ with $\psi(x)$ being the mother wavelet , i.e.,

$$\langle\psi|\Delta_h(q,p,\kappa)|\psi\rangle = \frac{e^{-\frac{p^2}{\kappa}}}{\sqrt{\pi\kappa}}\left|\int_{-\infty}^{\infty} dx\,\psi^*\left(\frac{x-s}{\mu}\right)e^{-x^2/2}\right|^2, \tag{1}$$

where $s = \frac{-1}{\sqrt{\kappa}}(\kappa q + ip)$, $\mu = \sqrt{\kappa}$, and $\Delta_h(q,p,\kappa)$ is introduced by Fan and Yang [12]

$$\begin{aligned}\Delta_h(q,p,\kappa) &= 2\int_{-\infty}^{\infty} dq'dp'\Delta_w(q',p')\exp\left[-\kappa(q'-q)^2 - \frac{(p'-p)^2}{\kappa}\right] \\ &= \frac{2\sqrt{\kappa}}{1+\kappa} : \exp\left\{\frac{-\kappa(q-Q)^2}{1+\kappa} - \frac{(p-P)^2}{1+\kappa}\right\} : ,\end{aligned} \tag{2}$$

where $\Delta_w(q,p)$ is the usual Wigner operator, $: \ :$ denotes normal ordering, $Q = (a+a^\dagger)/\sqrt{2}$ and $P = (a-a^\dagger)/(\sqrt{2}\mathtt{i})$ are the coordinate and the momentum operator, and $a_1, a_1^\dagger$ the Bose annihilation and creation operators, $[a, a^\dagger] = 1, a|0\rangle = 0$. $\Delta_h(q,p,\kappa)$ has been named the Husimi Wigner operator after Husimi, since $\langle\psi|\Delta_h(q,p)|\psi\rangle$ is just the Husimi distribution function,

$$\langle\psi|\Delta_h(q,p,\kappa)|\psi\rangle = 2\int_{-\infty}^{\infty} dq'dp' F_w(q',p')\exp\left[-\kappa(q'-q)^2 - \frac{(p'-p)^2}{\kappa}\right], \tag{3}$$

where $F_w(q',p') = \langle\psi|\Delta_h(q,p)|\psi\rangle$ is the usual Wigner function of $|\psi\rangle$. Equation (3) provides us with a convenient approach for calculating various Husimi distribution functions of miscellaneous quantum states.

Later in Ref. [13], Fan and Guo have introduced so-called entangled Husimi operator $\Delta_h(\sigma,\gamma,\kappa)$ which is endowed with definite physical meaning, and find the two-mode squeezed coherent state $|\sigma,\gamma\rangle_\kappa$ representation of $\Delta_h(\sigma,\gamma,\kappa)$, $\Delta_h(\sigma,\gamma,\kappa) = |\sigma,\gamma,\kappa\rangle\langle\sigma,\gamma,\kappa|$. The entangled Husimi operator $\Delta_h(\sigma,\gamma,\kappa)$ and the entangled Husimi distribution $F_h(\sigma,\gamma,\kappa)$ of quantum state $|\psi\rangle$ are given by

$$\Delta_h(\sigma,\gamma,\kappa) = 4\int d^2\sigma' d^2\gamma' \Delta_w(\sigma',\gamma')\exp\left\{-\kappa|\sigma'-\sigma|^2 - \frac{1}{\kappa}|\gamma'-\gamma|^2\right\}, \tag{4}$$

and

$$F_h(\sigma,\gamma,\kappa) = 4\int d^2\sigma' d^2\gamma' F_w(\sigma',\gamma')\exp\left\{-\kappa|\sigma'-\sigma|^2 - \frac{1}{\kappa}|\gamma'-\gamma|^2\right\}, \tag{5}$$

respectively, where $F_w(\sigma',\gamma') = \langle\psi|\Delta_w(\sigma',\gamma')|\psi\rangle$ is two-mode Wigner function, with $\Delta_w(\sigma',\gamma')$ being the two-mode Wigner operator. Thus we are naturally led to studying the entangled Husimi distribution function from the viewpoint of wavelet transformation.

In this paper, we shall extend the relation between wavelet transformation and Wigner-Husimi distribution function to the entangled case, that is to say, we employ the complex wavelet transformation (CWT) to investigate the entangled Husimi distribution function (EHDF) by bridging the relation between CWT and EHDF. We prove that the entangled Husimi distribution function of a two-mode quantum state $|\psi\rangle$ is just the modulus square of the complex wavelet transform of $e^{-|\eta|^2/2}$ with $\psi(\eta)$ being the mother wavelet up to a Gaussian function. Thus we present a convenient approach for calculating various entangled Husimi distribution functions of miscellaneous two-mode quantum states.

2 Complex Wavelet Transform and Its Quantum Mechanical Version

In Ref. [14], Fan and Lu have proposed the complex wavelet transform (CWT), i.e., the CWT of a complex signal function $g(\eta)$ by ψ is defined by

$$W_\psi g(\mu, z) = \frac{1}{\mu}\int \frac{d^2\eta}{\pi} g(\eta)\psi^*\left(\frac{\eta - z}{\mu}\right), \tag{6}$$

whose admissibility condition for mother wavelets, $\int \frac{d^2\eta}{2\pi}\psi(\eta) = 0$, is examined in the entangled state representations $\langle\eta|$ and a family of new mother wavelets (named the Laguerre–Gaussian wavelets) are found to match the CWT [14]. In fact, by introducing the bipartite entangled state representation $\langle\eta = \eta_1 + \mathrm{i}\eta_2|$, [15, 16]

$$|\eta\rangle = \exp\left\{-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right\}|00\rangle, \tag{7}$$

which is the common eigenvector of relative coordinate $Q_1 - Q_2$ and the total momentum $P_1 + P_2$,

$$(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \quad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle, \tag{8}$$

where Q_j and P_j are the coordinate and the momentum operator, related to the Bose operators $(a_j, a_j^\dagger)$, by $Q_j = (a_j + a_j^\dagger)/\sqrt{2}$ and $P_j = (a - a^\dagger)/(\sqrt{2}\mathrm{i})$ $(j = 1, 2)$, $[a_i, a_j^\dagger] = \delta_{ij}$, we can treat (5) quantum mechanically,

$$W_\psi g(\mu, z) = \frac{1}{\mu}\int \frac{d^2\eta}{\pi}\left\langle\psi\right|\left.\frac{\eta - z}{\mu}\right\rangle\langle\eta|g\rangle = \langle\psi|U_2(\mu, z)|g\rangle, \tag{9}$$

where $z = z_1 + iz_2 \in C$, $0 < \mu \in R$, $g(\eta) \equiv \langle\eta|g\rangle$ and $\psi(\eta) = \langle\eta|\psi\rangle$ are the wavefunction of state vector $|g\rangle$ and the mother wavelet state vector $|\psi\rangle$ in $\langle\eta|$ representation, respectively, and

$$U_2(\mu, z) \equiv \frac{1}{\mu}\int \frac{d^2\eta}{\pi}\left|\frac{\eta - z}{\mu}\right\rangle\langle\eta|, \quad \mu = e^\lambda, \tag{10}$$

is the two-mode squeezing-displacing operator [17–19]. Noticing that the two-mode squeezing operator has its natural expression in $\langle\eta|$ representation [16],

$$S_2(\mu) = \exp[(a_1^\dagger a_2^\dagger - a_1 a_2)\ln\mu] = \frac{1}{\mu}\int \frac{d^2\eta}{\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|, \tag{11}$$

which is different from the direct product of two single-mode squeezing (dilation) operators, and the two-mode squeezed state is simultaneously an entangled state, thus we can put (10) into the following form,

$$U_2(\mu, z) = S_2(\mu)\mathfrak{D}(z), \tag{12}$$

where $\mathfrak{D}(z)$ is a two-mode displacement operator, $\mathfrak{D}(z)|\eta\rangle = |\eta - z\rangle$ and

$$\begin{aligned}\mathfrak{D}(z) &= \int \frac{d^2\eta}{\pi}|\eta - z\rangle\langle\eta| \\ &= \exp\left[iz_1\frac{P_1 - P_2}{\sqrt{2}} - iz_2\frac{Q_1 + Q_2}{\sqrt{2}}\right] \\ &= D_1(-z/2)D_2(z^*/2).\end{aligned} \tag{13}$$

It the follows the quantum mechanical version of CWT is

$$W_\psi g(\mu, \zeta) = \langle\psi|S_2(\mu)\mathfrak{D}(z)|g\rangle = \langle\psi|S_2(\mu)D_1(-z/2)D_2(z^*/2)|g\rangle. \tag{14}$$

Equation (14) indicates that the CWT can be put into a matrix element in the $\langle\eta|$ representation of the two-mode displacing and the two-mode squeezing operators in (11) between the mother wavelet state vector $|\psi\rangle$ and the state vector $|g\rangle$ to be transformed. Thus the CWT differs from the direct product of two 1-dimensional wavelet transformations.

Once the state vector $\langle\psi|$ as mother wavelet is chosen, for any state $|g\rangle$ the matrix element $\langle\psi|U_2(\mu, z)|g\rangle$ is just the wavelet transform of $g(\eta)$ with respect to $\langle\psi|$. Therefore, various quantum optical field states can then be analyzed by their wavelet transforms.

3 Relation Between CWT and EHDF

In the following we shall show that the entangled Husimi distribution function (EHDF) of a quantum state $|\psi\rangle$ can be obtained by making a complex wavelet transform of the Gaussian function $e^{-|\eta|^2/2}$, i.e.,

$$\langle\psi|\Delta_h(\sigma, \gamma, \kappa)|\psi\rangle = e^{-\frac{1}{\kappa}|\gamma|^2}\left|\int \frac{d^2\eta}{\sqrt{\kappa}\pi}e^{-|\eta|^2/2}\psi^*\left(\frac{\eta - z}{\sqrt{\kappa}}\right)\right|^2, \tag{15}$$

where $\mu = e^\lambda = \sqrt{\kappa}$, $z = z_1 + iz_2$, and

$$z_1 = \frac{\cosh\lambda}{1+\kappa}[\gamma^* - \gamma - \kappa(\sigma^* + \sigma)], \tag{16}$$

$$z_2 = \frac{i\cosh\lambda}{1+\kappa}[\gamma + \gamma^* + \kappa(\sigma - \sigma^*)], \tag{17}$$

$\Delta_h(\sigma, \gamma, \kappa)$ is named the entangled Husimi operator by us,

$$\begin{aligned}\Delta_h(\sigma, \gamma, \kappa) = \frac{4\kappa}{(1+\kappa)^2} :\ \exp\Bigg\{&-\frac{(a_1 + a_2^\dagger - \gamma)(a_1^\dagger + a_2 - \gamma^*)}{1+\kappa} \\ &- \frac{\kappa(a_1 - a_2^\dagger - \sigma)(a_1^\dagger - a_2 - \sigma^*)}{1+\kappa}\Bigg\} : ,\end{aligned} \tag{18}$$

here $:\ :$ denotes normal ordering of operators. $\langle\psi|\Delta_h(\sigma, \gamma, \kappa)|\psi\rangle$ is the Husimi distribution function.

Proof of (15). When the state to be transformed is $|g\rangle = |00\rangle$ (the two-mode vacuum state), by noticing that $\langle\eta|00\rangle = e^{-|\eta|^2/2}$, we can express (9) as

$$\frac{1}{\mu}\int\frac{d^2\eta}{\pi}e^{-|\eta|^2/2}\psi^*\left(\frac{\eta - z}{\mu}\right) = \langle\psi|U_2(\mu, z)|00\rangle. \tag{19}$$

To combine the CWTs with transforms of quantum states more tightly and clearly, using the technique of integration within an ordered product (IWOP) [20–23] of operators, we can directly perform the integral in (10) [24]

$$\begin{aligned} U_2(\mu, z) &= \frac{1}{\mu}\int\frac{d^2\eta}{\pi} : \exp\left\{-\frac{\mu^2+1}{2\mu^2}|\eta|^2 + \frac{\eta z^* + z\eta^*}{2\mu^2} + \frac{\eta - z}{\mu}a_1^\dagger\right. \\ &\quad \left. - \frac{\eta^* - z^*}{\mu}a_2^\dagger + a_1^\dagger a_2^\dagger + \eta^* a_1 - \eta a_2 + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2 - \frac{|z|^2}{2\mu^2}\right\} : \\ &= \operatorname{sech}\lambda \exp\left[-\frac{1}{2(1+\mu^2)}|z|^2 + a_1^\dagger a_2^\dagger \tanh\lambda + \frac{1}{2}(z^* a_2^\dagger - z a_1^\dagger)\operatorname{sech}\lambda\right] \\ &\quad \times \exp[(a_1^\dagger a_1 + a_2^\dagger a_2)\ln\operatorname{sech}\lambda]\exp\left(\frac{z^* a_1 - z a_2}{1+\mu^2} - a_1 a_2\tanh\lambda\right), \end{aligned} \tag{20}$$

where we have set $\mu = e^\lambda$, $\operatorname{sech}\lambda = \frac{2\mu}{1+\mu^2}$, $\tanh\lambda = \frac{\mu^2-1}{\mu^2+1}$, and we have used the operator identity $e^{ga^\dagger a} =: \exp[(e^g - 1)a^\dagger a] :$. In particular, when $z = 0$, $U_2(\mu, z = 0)$ becomes to the usual normally ordered two-mode squeezing operator $S_2(\mu)$. From (20) it then follows that

$$\begin{aligned} U_2(\mu, z)|00\rangle &= \operatorname{sech}\lambda \exp\left\{-\frac{(z_1 - iz_2)(z_1 + iz_2)}{2(1+\mu^2)} + a_1^\dagger a_2^\dagger \tanh\lambda\right. \\ &\quad \left. + \frac{1}{2}[(z_1 - iz_2)a_2^\dagger - (z_1 + iz_2)a_1^\dagger]\operatorname{sech}\lambda\right\}|00\rangle. \end{aligned} \tag{21}$$

Substituting (16), (17) and $\tanh\lambda = \frac{\kappa-1}{\kappa+1}$, $\cosh\lambda = \frac{1+\kappa}{2\sqrt{\kappa}}$ into (21) yields

$$\begin{aligned} & e^{-\frac{1}{2\kappa}|\gamma|^2 - \frac{\sigma\gamma^* - \gamma\sigma^*}{2(\kappa+1)}}U_2(\mu, z_1, z_2)|00\rangle \\ &\quad = \frac{2\sqrt{\kappa}}{1+\kappa}\exp\left\{-\frac{|\gamma|^2 + \kappa|\sigma|^2}{2(\kappa+1)} + \frac{\kappa\sigma + \gamma}{1+\kappa}a_1^\dagger\right. \\ &\quad\quad \left. + \frac{\gamma^* - \kappa\sigma^*}{1+\kappa}a_2^\dagger + a_1^\dagger a_2^\dagger\frac{\kappa - 1}{\kappa + 1}\right\}|00\rangle \equiv |\sigma, \gamma\rangle_\kappa, \end{aligned} \tag{22}$$

then the CWT of (19) can be further expressed as

$$e^{-\frac{1}{2\kappa}|\gamma|^2 - \frac{\sigma\gamma^* - \gamma\sigma^*}{2(\kappa+1)}}\int\frac{d^2\eta}{\mu\pi}e^{-|\eta|^2/2}\psi^*\left(\frac{\eta - z_1 - iz_2}{\mu}\right) = \langle\psi|\sigma, \gamma\rangle_\kappa. \tag{23}$$

Using normally ordered form of the vacuum state projector $|00\rangle\langle 00| =: e^{-a_1^\dagger a_1 - a_2^\dagger a_2}:$, and the IWOP method as well as (22) we have

$$\begin{aligned}|\sigma,\gamma\rangle_{\kappa\kappa}\langle\sigma,\gamma| &= \frac{4\kappa}{(1+\kappa)^2}:\ \exp\bigg[-\frac{|\gamma|^2+\kappa|\sigma|^2}{\kappa+1}+\frac{\kappa\sigma+\gamma}{1+\kappa}a_1^\dagger+\frac{\gamma^*-\kappa\sigma^*}{1+\kappa}a_2^\dagger \\ &\quad +\frac{\kappa\sigma^*+\gamma^*}{1+\kappa}a_1+\frac{\gamma-\kappa\sigma}{1+\kappa}a_2+\frac{\kappa-1}{\kappa+1}(a_1^\dagger a_2^\dagger+a_1a_2)-a_1^\dagger a_1-a_2^\dagger a_2\bigg]: \\ &= \frac{4\kappa}{(1+\kappa)^2}:\ \exp\bigg\{-\frac{(a_1+a_2^\dagger-\gamma)(a_1^\dagger+a_2-\gamma^*)}{1+\kappa} \\ &\quad -\frac{\kappa(a_1-a_2^\dagger-\sigma)(a_1^\dagger-a_2-\sigma^*)}{1+\kappa}\bigg\}:\ =\Delta_h(\sigma,\gamma,\kappa).\end{aligned} \tag{24}$$

Now we explain why $\Delta_h(\sigma,\gamma,\kappa)$ is the entangled Husimi operator. Using the formula for converting an operator A into its Weyl ordering form [25]

$$A = 4\int\frac{d^2\alpha d^2\beta}{\pi^2}\langle-\alpha,-\beta|A|\alpha,\beta\rangle \,\vdots\, \exp\{2(\alpha^* a_1 - a_1^\dagger\alpha+\beta^* a_2 - a_2^\dagger\beta + a_1^\dagger a_1 + a_2^\dagger a_2)\}\,\vdots\,, \tag{25}$$

where the symbol $\vdots\ \vdots$ denotes the Weyl ordering, $|\beta\rangle$ is the usual coherent state, substituting (24) into (25) and performing the integration by virtue of the technique of integration within a Weyl ordered product of operators, we obtain

$$\begin{aligned}&|\sigma,\gamma\rangle_{\kappa\kappa}\langle\sigma,\gamma| \\ &\quad=\frac{16\kappa}{(1+\kappa)^2}\int\frac{d^2\alpha d^2\beta}{\pi^2}\langle-\alpha,-\beta|:\ \exp\bigg\{-\frac{(a_1+a_2^\dagger-\gamma)(a_1^\dagger+a_2-\gamma^*)}{1+\kappa} \\ &\qquad -\frac{\kappa(a_1-a_2^\dagger-\sigma)(a_1^\dagger-a_2-\sigma^*)}{1+\kappa}\bigg\}:|\alpha,\beta\rangle \\ &\qquad \times \,\vdots\, \exp\{2(\alpha^* a_1 - a_1^\dagger\alpha+\beta^* a_2 - a_2^\dagger\beta + a_1^\dagger a_1 + a_2^\dagger a_2)\}\,\vdots \\ &\quad = 4\,\vdots\,\exp\bigg\{-\kappa(a_1-a_2^\dagger-\sigma)(a_1^\dagger-a_2-\sigma^*)-\frac{1}{\kappa}(a_1+a_2^\dagger-\gamma)(a_1^\dagger+a_2-\gamma^*)\bigg\}\,\vdots\,,\end{aligned} \tag{26}$$

where we have used the integral formula

$$\int\frac{d^2z}{\pi}\exp(\zeta|z|^2+\xi z+\eta z^*) = -\frac{1}{\zeta}e^{-\frac{\xi\eta}{\zeta}}, \quad \mathrm{Re}(\zeta)<0. \tag{27}$$

This is the Weyl ordering form of $|\sigma,\gamma\rangle_{\kappa\kappa}\langle\sigma,\gamma|$. Then according to Weyl quantization scheme [26, 27] we know the Weyl ordering form of two-mode Wigner operator is given by

$$\Delta_w(\sigma,\gamma) = \,\vdots\,\delta(a_1-a_2^\dagger-\sigma)\delta(a_1^\dagger-a_2-\sigma^*)\delta(a_1+a_2^\dagger-\gamma)\delta(a_1^\dagger+a_2-\gamma^*)\,\vdots\,, \tag{28}$$

thus the classical corresponding function of a Weyl ordered operator is obtained by just replacing $a_1 - a_2^\dagger \to \sigma', a_1 + a_2^\dagger \to \gamma'$, i.e.,

$$\begin{aligned}
&4 \mathop{\vdots} \exp\left\{-\kappa(a_1 - a_2^\dagger - \sigma)(a_1^\dagger - a_2 - \sigma^*) - \frac{1}{\kappa}(a_1 + a_2^\dagger - \gamma)(a_1^\dagger + a_2 - \gamma^*)\right\} \mathop{\vdots} \\
&\quad \to 4 \exp\left\{-\kappa|\sigma' - \sigma|^2 - \frac{1}{\kappa}|\gamma' - \gamma|^2\right\},
\end{aligned} \tag{29}$$

and in this case the Weyl rule is expressed as

$$\begin{aligned}
|\sigma, \gamma\rangle_{\kappa\kappa}\langle\sigma, \gamma| &= 4 \int d^2\sigma' d^2\gamma' \mathop{\vdots} \delta(a_1 - a_2^\dagger - \sigma)\delta(a_1^\dagger - a_2 - \sigma^*)\delta(a_1 + a_2^\dagger - \gamma) \\
&\qquad \times \delta(a_1^\dagger + a_2 - \gamma^*) \mathop{\vdots} \exp\left\{-\kappa|\sigma' - \sigma|^2 - \frac{1}{\kappa}|\gamma' - \gamma|^2\right\} \\
&= 4 \int d^2\sigma' d^2\gamma' \Delta_w(\sigma', \gamma') \exp\left\{-\kappa|\sigma' - \sigma|^2 - \frac{1}{\kappa}|\gamma' - \gamma|^2\right\}.
\end{aligned} \tag{30}$$

In reference to (5) in which the relation between the entangled Husimi function and the two-mode Wigner function is shown, we know that the right-hand side of (30) should be just the entangled Husimi operator, i.e.

$$\begin{aligned}
|\sigma, \gamma\rangle_{\kappa\kappa}\langle\sigma, \gamma| &= 4 \int d^2\sigma' d^2\gamma' \Delta_w(\sigma', \gamma') \exp\left\{-\kappa|\sigma' - \sigma|^2 - \frac{1}{\kappa}|\gamma' - \gamma|^2\right\} \\
&= \Delta_h(\sigma, \gamma, \kappa),
\end{aligned} \tag{31}$$

thus (15) is proved by combining (31) and (23). Equation (31) can also be checked another way (see Appendix). □

Motivated by the preceding paper [11], we have further extended the relation between wavelet transformation and Wigner-Husimi distribution function to the entangled case. That is to say, we prove that the entangled Husimi distribution function of a two-mode quantum state $|\psi\rangle$ is just the modulus square of the complex wavelet transform of $e^{-|\eta|^2/2}$ with $\psi(\eta)$ being the mother wavelet up to a Gaussian function, i.e.,

$$\langle\psi|\Delta_h(\sigma, \gamma, \kappa)|\psi\rangle = e^{-\frac{1}{\kappa}|\gamma|^2} \left| \int \frac{d^2\eta}{\sqrt{\kappa}\pi} e^{-|\eta|^2/2} \psi^*((\eta - z)/\sqrt{\kappa}) \right|^2.$$

Thus we have a convenient approach for calculating various entangled Husimi distribution functions of miscellaneous quantum states. For more discussion about the wavelet transformation in the context of quantum optics, we refer to Refs. [28, 29].

Acknowledgements Work supported by the National Natural Science Foundation of China under grants. 10775097 and 10874174, and the Research Foundation of the Education Department of Jiangxi Province of China under Grant GJJ10097.

Appendix

We can check (31) by the following way.

Using the normally ordered form of the two-mode Wigner operator [13]

$$\Delta_w(\sigma,\gamma) = \frac{1}{\pi^2} : \exp\{-(a_1 - a_2^\dagger - \sigma)(a_1^\dagger - a_2 - \sigma^*) \\ - (a_1 + a_2^\dagger - \gamma)(a_1^\dagger + a_2 - \gamma^*)\} : , \tag{32}$$

we can further perform the integration in (4) and see

$$\begin{aligned}\Delta_h(\sigma,\gamma,\kappa) &= 4\int \frac{d^2\sigma' d^2\gamma'}{\pi^2} \exp\left\{-\kappa|\sigma'-\sigma|^2 - \frac{1}{\kappa}|\gamma'-\gamma|^2\right\} \\ &\quad \times : \exp\{-(a_1 - a_2^\dagger - \sigma)(a_1^\dagger - a_2 - \sigma^*) - (a_1 + a_2^\dagger - \gamma)(a_1^\dagger + a_2 - \gamma^*)\} : \\ &= \frac{4\kappa}{(1+\kappa)^2} : \exp\left\{-\frac{(a_1 + a_2^\dagger - \gamma)(a_1^\dagger + a_2 - \gamma^*)}{1+\kappa} \right. \\ &\quad \left. - \frac{\kappa(a_1 - a_2^\dagger - \sigma)(a_1^\dagger - a_2 - \sigma^*)}{1+\kappa}\right\} : \\ &= (24) = \Delta_h(\sigma,\gamma,\kappa),\end{aligned} \tag{33}$$

which is the confirmation of (31).

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